Nash Social Welfare Approximation for Strategic Agents

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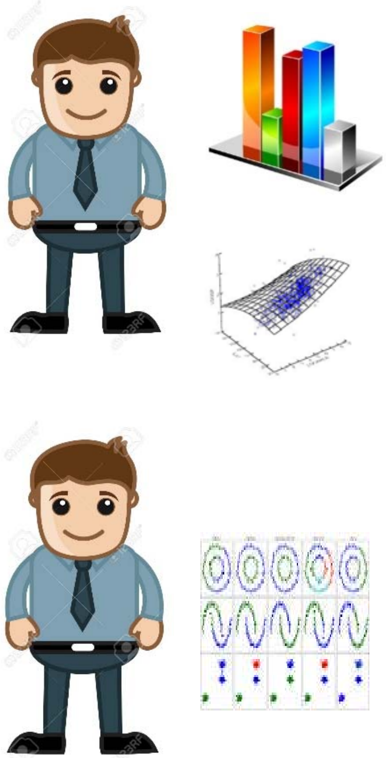
Resource Allocation

Optimal? Fair?
Stable?

Agents



⋮

$$V_i : R_+^m \rightarrow R_+$$

Scale invariant

Resources

m

…

Optimal allocation of resources to agents

Resource Allocation

Max Welfare

$\sum_i V_i$

Max Min

$\min_i V_i$

Solution changes if a V_i is scaled

Advantage to higher V_is

For one, many may suffer

Resource Allocation

Max Welfare

$$\sum_i \frac{V_i}{n}$$

Max Nash SW

$$\left(\prod_i V_i\right)^{\frac{1}{n}}$$

Max Min

$$\min_i \frac{V_i}{n}$$

$\rho = 1$

$\rho = 0$

$\rho = -\infty$

$$\left(\sum_i \frac{1}{n} V_i^{\rho}\right)^{\frac{1}{\rho}}$$

Generalized power mean

(n=# Agents)

Resource Allocation

Max Welfare

$$\sum_i \frac{V_i}{n}$$

Max Nash SW

$$\left(\prod_i V_i\right)^{\frac{1}{n}}$$

Max Min

$$\min_i V_i$$

Fair

- Scale invariant
- Envy-free (prefers own allocation than other’s)
- Proportional fair (fair share)

CEEI: competitive equilibrium with equal income (Varian’74)

Resource Allocation

Max Welfare

$\sum_i w_i V_i$

Max Nash SW

$\left(\Pi_i V_i^{w_i}\right)^{\frac{1}{\sum_i w_i}}$

NE

Max Min

$\min_i w_i V_i$

Fair
- Scale invariant
- Envy-free (don't want other's allocation)
- Proportional fair (fair share)

CEEI: competitive equilibrium with equal income (Varian'74)

Fisher Market

Goods

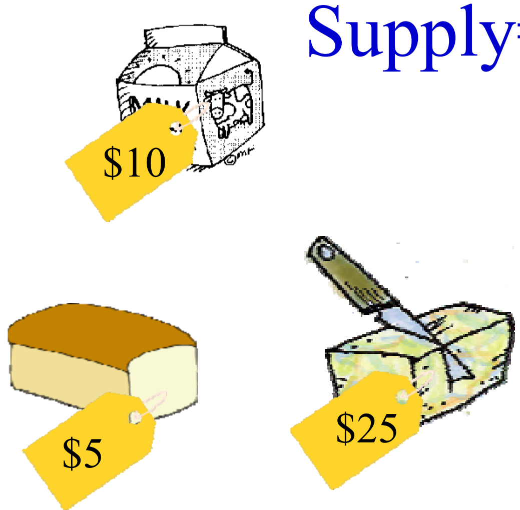
Divisible

Buyers

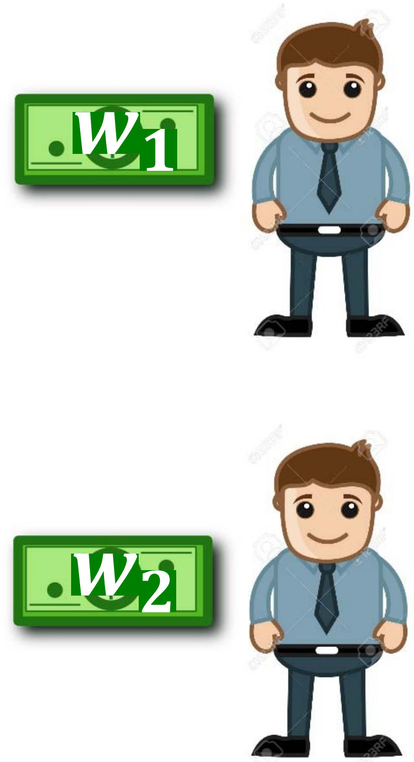
Amount of time used

Fisher Market

Supply=1



Goods (G)



V_i: Concave function

Buyers

Competitive (Market) Equilibrium:

Supply = Demand

Buyer i: Buys a bundle using money w_i so that V_i is maximized.

Market with Homogeneous Utilities

(Eisenberg'61)

$$\max \left(\Pi_i V_i(x_i)^{w_i}\right)^{\frac{1}{\sum_i w_i}}$$

Nash SW!

$s.t.$

$$\sum_i x_{ij} \leq 1, \qquad \forall j \in G$$

$$x_{ij} \geq 0, \qquad \forall (i,j)$$

Market with Homogeneous Utilities

(Eisenberg'61)

$$\max \sum_i w_i \log V_i(x)$$

$s.t.$

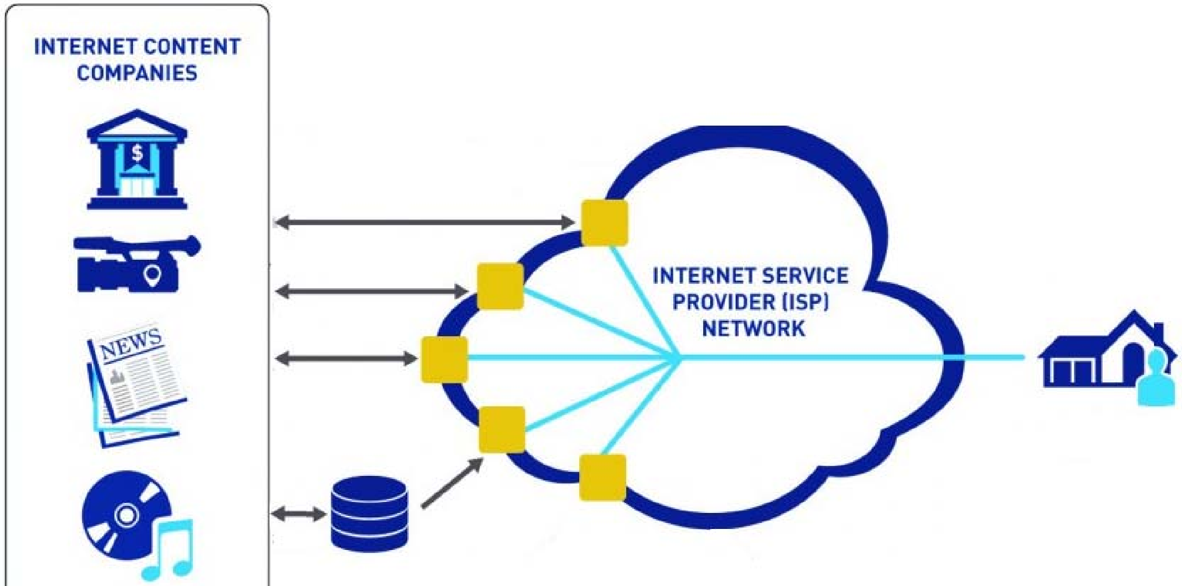
$$\sum_i x_{ij} \leq 1, \qquad \forall j \in G \longrightarrow p_j$$

$$x_{ij} \geq 0, \qquad \forall (i,j)$$

Dual var.

x_{ij} → Equilibrium allocation

p_j → Equilibrium prices



Stable? Optimal?
Fair?

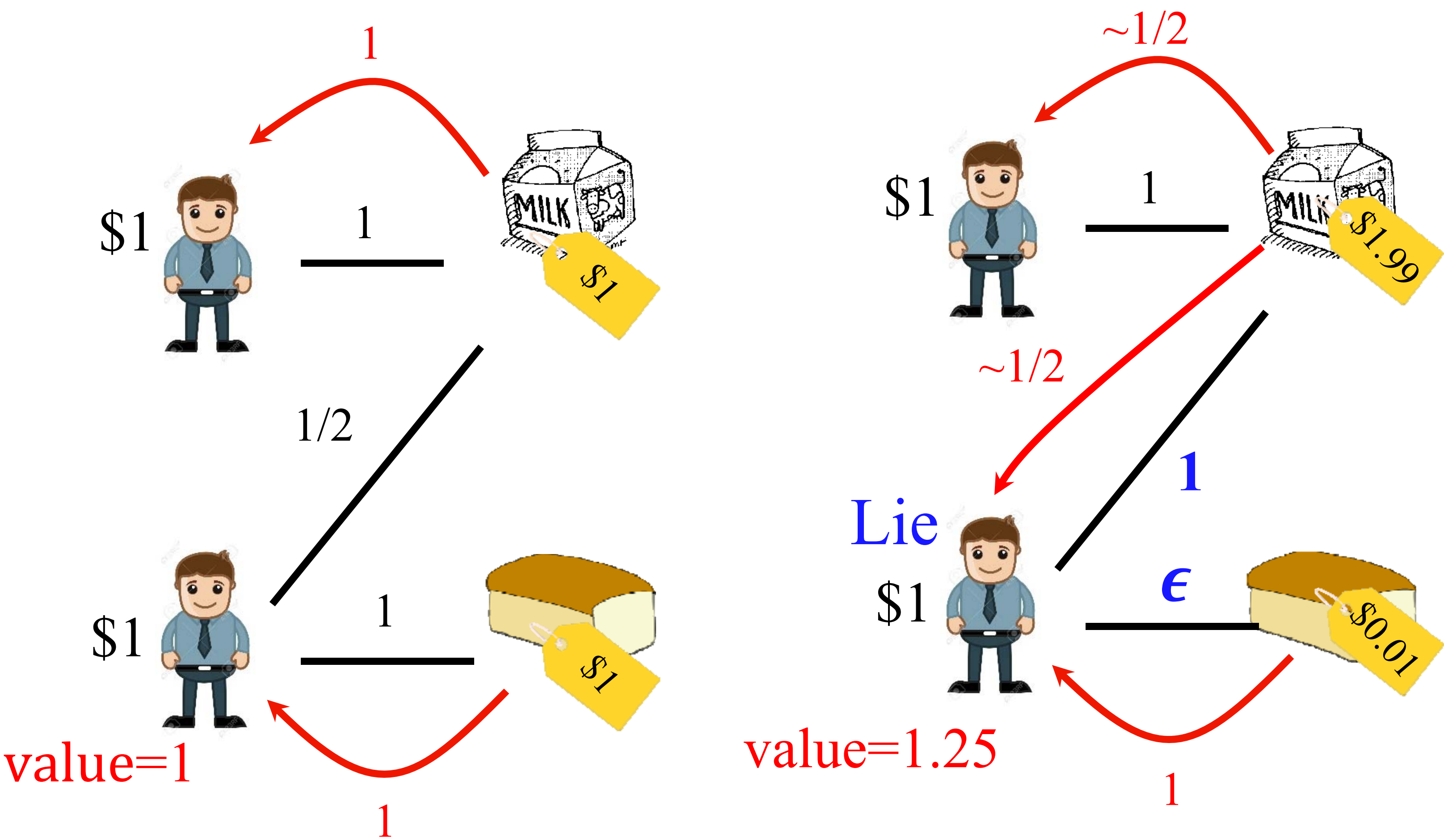
Markets ← Resource Allocation

↘ ↙

Nash Social Welfare (NSW)

Q: Would agents tell us their true V_i?

Additive utility



Lot of Work

- Nash equilibrium existence and analysis [AGBMS'11, MS'13, BLNP'14, BCDFFZ'14]
- Truthful mechanisms [MT'10, MN'12, CGG'13, CLPP'13, BM'15]
- Economics: Implementing competitive (market) equilibrium at a Nash equilibrium [DS'78, DHM'79, DG'03, …] – possible in large markets
- Fair-division [G'03, DFHKL'12, GZHKZZ'11, …]

Markets ← Resource Allocation

Nash Social Welfare (NSW)

Strategic Agents will lie about their V_i

Q: What will be the efficiency loss?

Price of Anarchy (PoA): $\max_{NE} \frac{OPT\ NSW}{NSW\ at\ Nash\ Equilibrium\ (NE)}$

NE: No unilateral deviation

We show

$V_1, \ldots, V_n$ → Fisher market mechanism → $p, (\overline{x_1}, \ldots, \overline{x_n})$

Additive:

$$V_i(x_i) = \sum_j v_{ij} x_{ij}$$

Perfect Substitutes

$$e^{\frac{1}{e}} \leq \text{PoA} \leq 2$$

Leontief:

$$V_i(x_i) = min_j \frac{x_{ij}}{v_{ij}}$$

Perfect Complements

$$n \leq \boxed{\text{PoA} \leq n}$$

Holds for general concave

Lemma. PoA $\leq$ n

Proof. Suppose $w_i = 1, \forall i$

Total prices $\leq$ total money $\leq n$

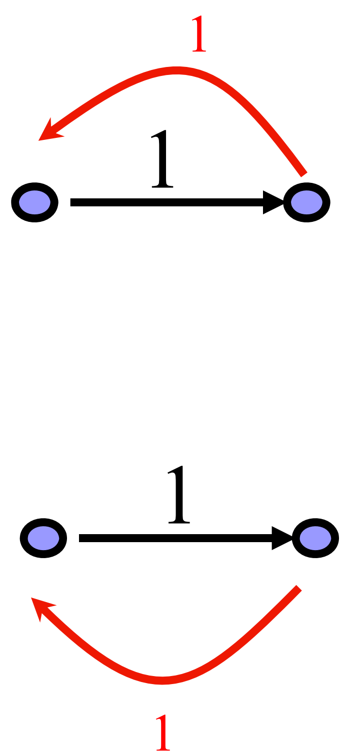
NE allocation $X = [x_{ij}]$ OPT allocation $X^* = [x^*_{ij}]$

$$V_i(x_i) \geq V_i\left(\frac{1}{n}, \dots, \frac{1}{n}\right) \geq \frac{V_i(1, \dots, 1)}{n} \geq V_i(x_i^*)$$

$$\frac{OPT\ NSW}{NE\ NSW} = \left(\Pi_i \frac{V_i(x_i^*)}{V_i(x_i)}\right)^{1/n} \leq (\pi_i n)^{\frac{1}{n}} = n$$

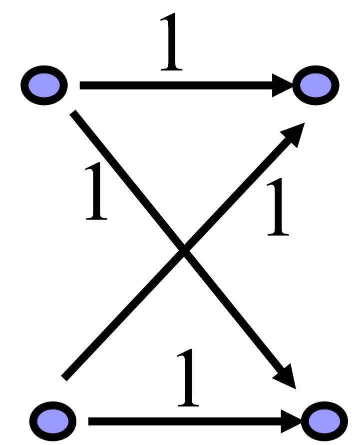
Lemma. PoA $\geq$ n for Leontief $\left(V_i(x_i) = \min_j \frac{x_{ij}}{v_{ij}}\right)$

Proof. Suppose $w_i = 1, \forall i$



NSW=1

NE

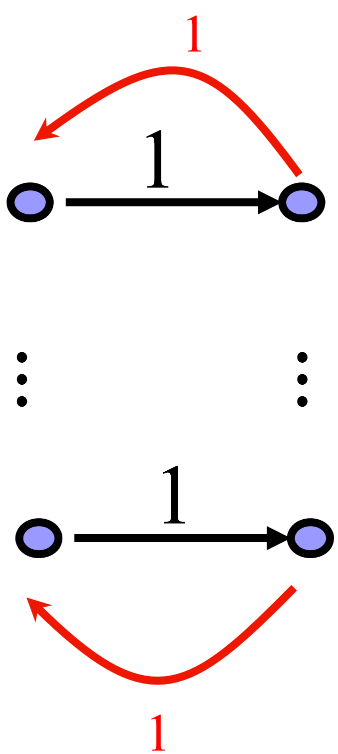


$x_{ij} = \frac{1}{2}$

NSW= $\frac{1}{2}$

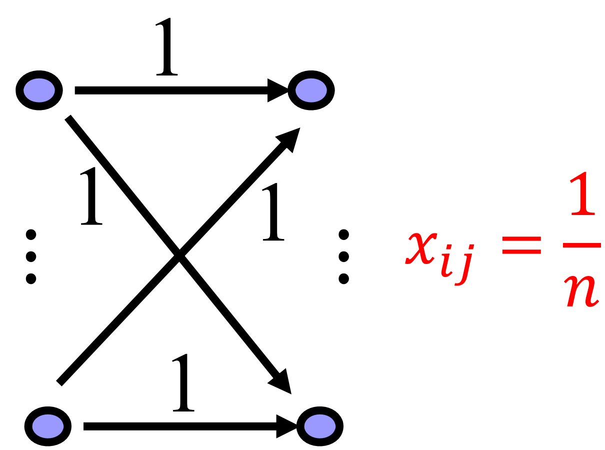
Lemma. PoA $\geq$ n for Leontief $\left(V_i(x_i) = \min_j \frac{x_{ij}}{v_{ij}}\right)$

Proof. Suppose $w_i = 1, \forall i$



NSW=1

NE



NSW= $\frac{1}{n}$

Trading Post mechanism

Each agent i uses budget w_i to place bids for each good j.

After all the agents place their bids, agent i gets a fraction of good j proportional to her bid (and zero if they bid nothing); if the goods are indivisible → probability.

(shapley-shubik game, Chinese auction, proportional sharing, Tullock contest)

We show

i *bids* b_{ij} for good j → Trading-post mechanism → $p_j = \sum_i b_{ij}$, $x_{ij} = \frac{b_{ij}}{p_j}$

Arbitrary concave utility functions:

PoA ≤ 2

Leontief: If all positive prices at market equilibrium then PoA = 1.
Otherwise no Nash equilibrium.

We show

b_{ij}
0 or $> \delta$ → **Trading-post(δ) mechanism** → $p_j = \sum_i b_{ij}$, $x_{ij} = \frac{b_{ij}}{p_j}$

Leontief

Result 1:
TP(δ) has pure Nash equilibrium for every $\delta > 0$.

Result 2:
For every $\epsilon > 0$, there exists $\delta > 0$ such that for TP(δ), PoA $\leq 1 + \epsilon$.

Main message

Trading Post forces the agents to put their money where their mouth is $\rightarrow$ removes the bad equilibria of the Fisher market

⇓

Trading post is a much better mechanism in terms of *implementing* competitive equilibria in case of homogeneous valuations

Theorem. If utility functions are concave then PoA of TP mechanism is 2.

Proof. OPT allocation x_{ij}^* NE bids b_{ij}

Agent i: withdraw all money, and buy proportional to x_i^*

$$x_i' = x_i^*/\beta_i \ (\beta_i > 1)$$

$$V_i(x_i) \overset{\text{NE}}{\geq} V_i(x_i') = V_i\left(\frac{x_i^*}{\beta_i}\right) \overset{\text{concave}}{\geq} \frac{1}{\beta_i} V_i(x_i^*) \quad \Rightarrow \frac{V_i(x_i^*)}{V_i(x_i)} \leq \beta_i$$

… …

$$\sum_i w_i \beta_i \leq 2 \sum_i w_i$$

$$\frac{OPT\ NSW}{NSW\ at\ NE} \leq \left(\Pi_i \beta_i^{w_i}\right)^{\frac{1}{\sum_i w_i}} \underset{\text{AM-GM}}{\leq} \sum_i \frac{w_i}{\sum_i w_i} \beta_i \leq 2$$

Theorem. For every $\epsilon > 0$, there exists $\delta > 0$ s.t. NE of TP(δ), approximates OPT NSW within $(1 + \epsilon)$.

Proof. Take any Nash equilibrium of TP(δ)

1. Show that items received in higher fractions by any player i must have minimum bids δ by i (otherwise a lower bid would suffice)

2. Show that this gives ε-competitive equilibrium from these bids (all goods sold, all money spent, each player gets an ε-optimal bundle)

3. Show that ε-competitive equilibria are approximate solutions of Eisenberg's convex program for Leontief utilities ⇒ good NSW

Fairness Guarantees

- For both Fisher and Trading post mechanisms, utility of agents at Nash equilibria is weighted proportional

$$V_i(NE) \geq \frac{w_i}{\sum_i w_i} \mathrm{V_i}(\text{All the resources})$$

- Similar (approximate) guarantee holds for TP(δ)

Trading post

- No mixed Nash equilibria even in the general concave case.
  - Given opponents bids (even if randomized), valuation function of agent i in her bids is strictly concave → unique best response.

- Mixed NE PoA ≤ 2

Open Questions

- PoA at (coarse) correlated equilibria of Trading post?
  - Convergence points of no-regret dynamics
- Computation of Nash equilibria in Trading Post.
- Truthful non-wasteful mechanism with good approximation for NSW

THANK YOU